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On Abelian multi-Chern–Simons field theories

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Abstract

In this paper a class of multi-Chern–Simons field theories which is relevant to the statistical mechanics of polymer systems is investigated. Motivated by the problems which one encounters in the treatment of these theories, a general procedure is presented to eliminate the Chern–Simons fields from their action. In this way it has been possible to derive an expression of the partition function of topologically linked polymers which depends explicitly on the topological numbers and does not have intractable nonlocal terms as it happened in previous approaches. The new formulation of multi-Chern–Simons field theories is then used to remove and clarify some inconsistencies and ambiguities which apparently affect field theoretical models of topologically linked polymers. Finally, the limit of disentangled polymers is discussed.

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1. Foreword

In this paper we study Abelian multi-Chern–Simons field theories [1–5] coupled to charged scalar fields. Models of this kind find a natural application in the statistical mechanics of closed polymer rings subject to topological constraints [6] and are relevant in the phenomenology of the fractional quantum Hall effect [7]. An important feature of these theories is that they exhibit the phenomenon of charge confinement, which occurs via a topological mechanism explained in [7, 8]. The aim of this work is to solve some problems, discussed below, which arise in the field theoretical formulation of polymer systems.

Let us suppose that there are N polymers with trajectories P_1, \dots, P_N in a dilute solution. The topological state of the system is specified in such a way that each trajectory P_i winds up around trajectory P_j a number of times m_{ij} , $i > j = 1, \dots, N$. In principle, it would be desirable to write down an expression of the polymer partition function which depends on the Gauss linking numbers m_{ij} . In practice, however, it is only possible to compute the partition function in the space of the Fourier conjugated variables λ_{ij} [9]. All attempts to go back to the space of the real topological numbers m_{ij} with an inverse Fourier transformation have led

so far to field theories with intractable nonlocal terms [9, 11]. Even in the investigation of the simplest case in which all the polymers are disentangled, i.e. $m_{ij} = 0$ for $i, j = 1, \dots, N$, one encounters tremendous complications, which can be overcome only by resorting to mean-field-like approximations [10]. Moreover, it is also hard to give a physical meaning to the parameters λ_{ij} . At first sight, in multi-Chern–Simons polymer models they play the role of coupling constants and determine the ‘strength’ of the topological interactions which are necessary to keep the polymer trajectories in the given topological state. However, there is some freedom in the choice of the domain in which these parameters are defined, so that this interpretation cannot be correct. For example, each λ_{ij} can be defined in the interval $[-\pi, \pi]$, but any other interval of the kind $[l\pi, (l+2)\pi]$, where l is an arbitrary integer, is also allowed due to the properties of Fourier transformations. Clearly, from the point of view of field theories, it makes a great difference if $l = 1$ or if $l \gg 1$. Furthermore, also the range $(-\infty, +\infty)$ is justified if one starts from a model of open polymers and then recovers the limit of closed polymers requiring that the ends of the polymers coincide.

Unfortunately, it is not an easy task to obtain a reasonably simple expression of the polymer partition function in terms of the physical parameters m_{ij} and to remove the above ambiguities. One problem is that topological interactions in polymer systems are governed by Chern–Simons fields and the contribution of these fields to the partition function is hard to evaluate. First of all, it is not possible to proceed perturbatively, since the values of the ‘coupling constants’ λ_{ij} are not small. On the other hand, the action of Chern–Simons is intrinsically defined in three dimensions, so that it is difficult to exploit techniques such as the ϵ -expansion which require its extension to arbitrary dimensions [12]. Also from the point of view of numerical simulations the situation is no better, since the lattice formulation of Chern–Simons field theories has so far encountered many obstacles [13, 14].

In order to solve these problems, we rewrite the action of multi-Chern–Simons field theories with the help of a Hubbard–Stratonovich transformation and successively eliminate the Chern–Simons fields. The resulting field theories contain only scalar fields and a set of auxiliary fields which have no dynamics. The procedure used is quite general, although it has been developed for the special multi-Chern–Simons field theories which are relevant to polymers. Exploiting the new formulation of the polymer partition function, we are able to prove that there are no ambiguities in the field theoretical description of polymer systems, because all the possible domains in which the coupling constants $\lambda_1, \dots, \lambda_\Gamma$ may be defined lead to equivalent models. Finally, we clarify the physical meaning of these coupling constants: they play the role of Lagrange multipliers. The constraints which they impose are explicitly computed.

The material presented in this paper is divided as follows. In section 2 we consider an action with two Chern–Simons fields coupled with multiplets of charged scalar fields which are invariant under a $U(n_1) \times U(n_2)$ group of global symmetry. The relations of this model with the statistical mechanics of two topologically linked polymers have been discussed in detail in [6, 15] and are here only briefly summarized. Starting from this simple example of multi-Chern–Simons field theories, we illustrate the problems which arise in polymer models. In section 3 it is shown how it is possible to eliminate the Chern–Simons fields using a suitable Hubbard–Stratonovich transformation. In this way we obtain an expression of the partition function of polymers in terms of the topological numbers m_{ij} in which the action is polynomial in the fields. In previous approaches, instead, the action contained non-polynomial and nonlocal terms. In section 4 we check that the non-uniqueness of the domain of integration of the coupling constants disappears due to a symmetry which was hidden in the original formulation in terms of Chern–Simons fields. Finally, in sections 5

and 6 these results are generalized to the case of multi-Chern–Simons field theories describing the statistical mechanics of an arbitrary number of polymers.

2. The two-polymer model

Let us consider the action:

$$S(\lambda) = \int d^3x [i\kappa \mathbf{B} \cdot (\nabla \times \mathbf{A}) + |(\nabla - i\kappa \mathbf{B})\Psi_1|^2 + m_1^2 |\Psi_1|^2] + \int d^3x [|(\nabla - i\lambda \mathbf{A})\Psi_2|^2 + m_2^2 |\Psi_2|^2]. \tag{1}$$

In equation (1) $i = \sqrt{-1}$ and the symbols $\Psi_i, \Psi_i^*, i = 1, 2$, denote multiplets of charged fields:

$$\Psi_i = (\psi_i^1, \dots, \psi_i^{n_i}) \quad \Psi_i^* = (\psi_i^{*1}, \dots, \psi_i^{*n_i}). \tag{2}$$

In our notation

$$|(\nabla - i\kappa \mathbf{B})\Psi_1|^2 = \sum_{a=1}^{n_1} (\nabla + i\kappa \mathbf{B})\psi_1^{*a} \cdot (\nabla - i\kappa \mathbf{B})\psi_1^a \tag{3}$$

$$|(\nabla - i\kappa \mathbf{A})\Psi_2|^2 = \sum_{a=1}^{n_2} (\nabla + i\kappa \mathbf{A})\psi_2^{*a} \cdot (\nabla - i\kappa \mathbf{A})\psi_2^a \tag{4}$$

$$|\Psi_i|^2 = \sum_{a=1}^{n_i} \psi_i^{*a} \psi_i^a \quad i = 1, 2. \tag{5}$$

It is easy to check by rescaling the field \mathbf{B} that the Chern–Simons coupling constant κ is irrelevant and that only the coupling constant λ appears in the physical amplitudes of the theory.

Field theories such as those of equation (1) enter various physical problems. Here we consider the case of two closed polymers with trajectories P_1 and P_2 and lengths L_1 and L_2 , respectively. The trajectories are constrained to satisfy the following topological constraint:

$$\chi(P_1, P_2) = m \quad m = 0, \pm 1, \pm 2 \dots \tag{6}$$

where $\chi(P_1, P_2)$ is the Gauss linking number given by

$$\chi(P_1, P_2) = \frac{1}{4\pi} \int_0^{L_1} ds_1 \int_0^{L_2} ds_2 \dot{\mathbf{x}}_1(s_1) \cdot \left[\dot{\mathbf{x}}_2(s_2) \times \frac{(\mathbf{x}_1(s_1) - \mathbf{x}_2(s_2))}{|\mathbf{x}_1(s_1) - \mathbf{x}_2(s_2)|^3} \right]. \tag{7}$$

In the above equation P_1 and P_2 are represented by two closed curves $\mathbf{x}_1(s_2), \mathbf{x}_2(s_2)$, where s_1 and s_2 are the arc lengths of the trajectories. In terms of the bond currents

$$\mathbf{j}_i(\mathbf{x}) = \int_0^{L_i} ds_i \mathbf{x}_i(s_i) \delta^{(3)}(\mathbf{x} - \mathbf{x}_i(s_i)). \tag{8}$$

Equation (7) may be rewritten as follows:

$$m = \frac{1}{4\pi} \int d^3x d^3y \mathbf{j}_1(\mathbf{x}) \cdot \left[\mathbf{j}_2(\mathbf{y}) \times \frac{(\mathbf{x} - \mathbf{y})}{|\mathbf{x} - \mathbf{y}|^3} \right]. \tag{9}$$

It is possible to show that, in the Lorentz gauge, in which the fields \mathbf{A} and \mathbf{B} are completely transverse, the partition function of this two-polymer system coincides with the following amplitude [15, 16]:

$$Z = \int_0^{2\pi} \frac{d\lambda}{2\pi} e^{-im\lambda} Z(\lambda) \tag{10}$$

where

$$Z(\lambda) = \int \mathcal{D}(\text{fields}) |\psi_1^1(\mathbf{x})|^2 |\psi_2^1(\mathbf{y})|^2 e^{-S_2(\lambda)} \quad (11)$$

and

$$\mathcal{D}(\text{fields}) = \int \mathcal{D}\mathbf{A}\mathcal{D}\mathbf{B} \prod_{i=1}^2 \prod_{a_i=1}^{n_i} [\mathcal{D}\psi_i^{a_i} \mathcal{D}\psi_i^{*a_i}]. \quad (12)$$

To make contact with polymer physics, we should keep in mind that it is still necessary to continue analytically the partition function Z to the limit of zero replica numbers n_1 and n_2 . Moreover, one should also add to action (1) the so-called excluded volume interactions, which take into account the steric repulsions of the monomers. However, both zero replica limit and excluded volume interactions are irrelevant in the present context and will be ignored.

We see from equation (10) that one has to consider the sum over the partition functions $Z(\lambda)$ for all values of the coupling constant λ in the interval $[0, 2\pi]$. This is a consequence of the fact that the topological condition (6) has been imposed by inserting in the partition function the δ of Kronecker $\delta_{m,\chi(P_1, P_2)}$, which in the Fourier representation is given by

$$\delta_{m,\chi(P_1, P_2)} = \int_0^{2\pi} \frac{d\lambda}{2\pi} e^{-i\lambda(m-\chi(P_1, P_2))}. \quad (13)$$

This integration over λ is a further complication with respect to standard field theories, which makes it difficult to study the physical properties of the two-polymer model given above. For example, let us note that the right-hand side of equation (13) is invariant under the shifts:

$$\lambda \longrightarrow \lambda + \pi k \quad k = 0, \pm 1, \pm 2 \dots \quad (14)$$

Therefore, for consistency, the partition function Z should also be invariant as a function of λ under these shifts. However, such invariance is not evident from equations (10), (11) and from action (1). Other difficulties arise if we wish to describe the behaviour of two disentangled polymer rings starting from the partition function of equations (10), (11). As a matter of fact, even in the limit of zero topological number m , the integration over λ remains complicated. In general, the investigation of the $m = 0$ limit is problematic in models of topologically linked polymers based on the Edwards approach. The only concrete results have been achieved up to now in the case of dense solutions, where mean-field-like approximations are possible [10].

In principle, one can easily eliminate the coupling constant λ in (10) by performing a simple Gaussian integral, but the new partition function contains nonlocal operators whose treatment by analytical methods is difficult [9]. On the other hand, it is possible to study the partition function $Z(\lambda)$ by means of field theoretical techniques. However, knowledge of the properties of $Z(\lambda)$ does not provide a very deep insight into the properties of the final partition function Z .

3. Elimination of the Chern–Simons fields

A more transparent formulation of the two-polymer problem, in which the role of the coupling constant λ is explicit, can be provided by means of two Hubbard–Stratonovich transformations. As a first step, let us rewrite the partition function of equations (10), (11) in the following way:

$$Z = \int_0^{2\pi} \frac{d\lambda}{2\pi} e^{-im\lambda} \int \mathcal{D}\mathbf{A}\mathcal{D}\mathbf{B} \prod_{i=1}^2 \left[\prod_{a_i=1}^{n_i} \mathcal{D}\psi_i^{a_i} \mathcal{D}\psi_i^{*a_i} \prod_{b_i=1}^{n_i} \mathcal{D}\xi_i^{b_i} \mathcal{D}\xi_i^{*b_i} \right] \times |\psi_1^1(\mathbf{x})|^2 |\psi_2^1(\mathbf{y})|^2 e^{-S_0 - S_1(\lambda)} \quad (15)$$

where ξ_2^{*a} , ξ_2^a represent two sets of auxiliary complex vector fields and $S_0, S_1(\lambda)$ are given by

$$S_0 = \sum_{i=1}^2 \sum_{a_i=1}^{n_i} \int d^3x \{ -i [\nabla \psi_i^{*a_i} \cdot \xi_i^{a_i} + \nabla \psi_i^{a_i} \cdot \xi_i^{*a_i}] + m_i^2 |\psi_i^{a_i}|^2 + \xi_i^{a_i} \cdot \xi_i^{*a_i} \} \tag{16}$$

$$S_1(\lambda) = i\kappa \int d^3x \mathbf{B} \cdot (\nabla \times \mathbf{A}) + i\kappa \int d^3x \mathbf{B} \cdot \mathbf{J}_1 + i\lambda \int d^3x \mathbf{A} \cdot \mathbf{J}_2. \tag{17}$$

Here we have introduced the vector fields

$$\mathbf{J}_i = \frac{1}{i} \sum_{a_i=1}^{n_i} [\psi_i^{*a_i} \xi_i^{a_i} - \psi_i^{a_i} \xi_i^{*a_i}] \tag{18}$$

which are related to the total matter currents of the replica fields $\psi_i^{a_i}, \psi_i^{*a_i}$. This connection with matter currents becomes more explicit if we consider the classical equations of motion of the fields $\xi_i^{a_i}, \xi_i^{*a_i}$:

$$\xi_1^{a_1} = (\nabla - i\kappa \mathbf{B}) \psi_1^{a_1} \quad \xi_1^{*a_1} = (\nabla + i\kappa \mathbf{B}) \psi_1^{*a_1} \tag{19}$$

$$\xi_2^{a_2} = (\nabla - i\lambda \mathbf{A}) \psi_2^{a_2} \quad \xi_2^{*a_2} = (\nabla + i\lambda \mathbf{A}) \psi_2^{*a_2}. \tag{20}$$

Substituting equations (19) and (20) in (18) one obtains

$$\mathbf{J}_1 = \frac{1}{i} \sum_{a_1=1}^{n_1} [\psi_1^{*a_1} (\nabla - i\kappa \mathbf{B}) \psi_1^{a_1} - \psi_1^{a_1} (\nabla + i\kappa \mathbf{B}) \psi_1^{*a_1}] \tag{21}$$

$$\mathbf{J}_2 = \frac{1}{i} \sum_{a_2=1}^{n_2} [\psi_2^{*a_2} (\nabla - i\lambda \mathbf{A}) \psi_2^{a_2} - \psi_2^{a_2} (\nabla + i\lambda \mathbf{A}) \psi_2^{*a_2}] \tag{22}$$

which are exactly the total Abelian matter currents of the replica fields.

At this point we are ready to show that the partition function (10) and (15) are equivalent. To prove that, it is sufficient to perform in equation (15) the change of variables:

$$\xi_1^{a_1} = \xi_1'^{a_1} + i(\nabla - i\kappa \mathbf{B}) \psi_1^{a_1} \quad \xi_1^{*a_1} = \xi_1'^{*a_1} + i(\nabla + i\kappa \mathbf{B}) \psi_1^{*a_1} \tag{23}$$

$$\xi_2^{a_2} = \xi_2'^{a_2} + i(\nabla - i\lambda \mathbf{A}) \psi_2^{a_2} \quad \xi_2^{*a_2} = \xi_2'^{*a_2} + i(\nabla + i\lambda \mathbf{A}) \psi_2^{*a_2}. \tag{24}$$

After this substitution, the result is exactly the partition function of equations (10), (11), apart from an irrelevant constant coming from the Gaussian integration over the decoupled primed fields $\xi_i'^{a_i}, \xi_i'^{*a_i}, i = 1, 2, a_i = 1, \dots, n_i$. Hubbard–Stratonovich transformations of this kind are common in polymer physics [17, 18]. They can be rewritten in a more familiar form in terms of real vector fields, coinciding with the real and imaginary parts of the fields $\xi_i^{a_i}$ and $\xi_i^{*a_i}$. This point will be discussed in more detail in the appendix.

It is now easy to integrate out the Chern–Simons field from the partition function (15). To this purpose, we have to consider the path integral:

$$Z_{\psi\xi} = \int \mathcal{D}\mathbf{A} \mathcal{D}\mathbf{B} e^{-S_1(\lambda)}. \tag{25}$$

A first integration over the \mathbf{B} fields gives

$$Z_{\psi\xi} = \int \mathcal{D}\mathbf{A} e^{-\lambda \int d^3x \mathbf{A} \cdot \mathbf{J}_2} \delta(\nabla \times \mathbf{A} + \mathbf{J}_1). \tag{26}$$

The δ -function in equation (26) enforces the constraint:

$$\nabla \times \mathbf{A} + \mathbf{J}_1 = 0 \quad (27)$$

which implies that the current \mathbf{J}_1 is conserved as expected. As a matter of fact, taking the divergence of both members of equation (27), one obtains $\nabla \cdot \mathbf{J}_1 = 0$.

Solving the constraint (27) with respect to \mathbf{A} , equation (26) becomes

$$Z_{\psi\xi} = \exp \left[-i \frac{\lambda}{4\pi} \int d^3x d^3y \left(\nabla \frac{1}{|\mathbf{x} - \mathbf{y}|} \times \mathbf{J}_2(\mathbf{y}) \right) \cdot \mathbf{J}_1(\mathbf{x}) \right]. \quad (28)$$

The substitution of (28) in the partition function (15) gives the following result:

$$Z = \int_0^{2\pi} \frac{d\lambda}{2\pi} e^{-im\lambda} \int \mathcal{D}(\text{fields})' |\psi_1^1(\mathbf{x})|^2 |\psi_2^1(\mathbf{y})|^2 \times \exp \left[-S_0 - i \frac{\lambda}{4\pi} \int d^3x d^3y \left(\nabla \frac{1}{|\mathbf{x} - \mathbf{y}|} \times \mathbf{J}_2(\mathbf{y}) \right) \cdot \mathbf{J}_1(\mathbf{x}) \right]. \quad (29)$$

The field integration measure is now

$$\mathcal{D}(\text{fields})' = \prod_{i=1}^2 \prod_{a_i=1}^{n_i} \mathcal{D}\psi_i^{a_i} \mathcal{D}\psi_i^{*a_i} \mathcal{D}\xi_i^{a_i} \mathcal{D}\xi_i^{*a_i}. \quad (30)$$

4. The limit of disentangled polymers

With the new formulation of the partition function given by equation (29), the role of the parameter λ in the two-polymer model and its invariance under the shifts $\lambda \rightarrow \lambda + \pi$ have become transparent. As a matter of fact, performing the simple integration over λ one obtains

$$Z = \int \mathcal{D}(\text{fields})' |\psi_1^1(\mathbf{x})|^2 |\psi_2^1(\mathbf{y})|^2 e^{-S_0} \times \delta \left(m - \frac{1}{4\pi} \int d^3x d^3y \mathbf{J}_1(\mathbf{x}) \cdot \left(\mathbf{J}_2(\mathbf{y}) \times \nabla \frac{1}{|\mathbf{x} - \mathbf{y}|} \right) \right) \quad (31)$$

where the measure $\mathcal{D}(\text{fields})'$ and the action S_0 have been given in equations (30) and (16), respectively. Clearly, the constraint

$$\frac{1}{4\pi} \int d^3x d^3y \mathbf{J}_1(\mathbf{x}) \cdot \left(\mathbf{J}_2(\mathbf{y}) \times \nabla \frac{1}{|\mathbf{x} - \mathbf{y}|} \right) = m \quad (32)$$

is the analogue of the present field theoretical formalism of the topological constraint (9).

In the limit $m = 0$, the partition function (29) becomes

$$Z_{m=0} = \int \mathcal{D}(\text{fields})' |\psi_1^1(\mathbf{x})|^2 |\psi_2^1(\mathbf{y})|^2 e^{-S_0} \times \delta \left(\frac{1}{4\pi} \int d^3x d^3y \mathbf{J}_1(\mathbf{x}) \cdot \left(\mathbf{J}_2(\mathbf{y}) \times \nabla \frac{1}{|\mathbf{x} - \mathbf{y}|} \right) \right). \quad (33)$$

From equation (33) it turns out that the topological interactions do not vanish if $m = 0$ as one could naively expect from the fact that the polymers are disentangled in this case. The reason is that the topological interactions are still necessary when the polymers get too near at some point in order to prevent the crossing of the trajectories, which would modify the value of m .

Equation (29) also shows that the form of the partition function Z does not change under a shift of the coupling constant λ of the kind given in equation (14). As a matter of fact, let

us perform the shift $\lambda \rightarrow \lambda + \pi k = \lambda', k = \pm 1, \pm 2, \dots$ in equation (10), so that we get the new partition function:

$$Z_{\text{shifted}} = \int_{\pi k}^{\pi(k+1)} \frac{d\lambda'}{2\pi} e^{-im\lambda'} Z(\lambda') = (-1)^{km} \int_0^{2\pi} \frac{d\lambda}{2\pi} e^{-im\lambda} Z(\lambda + \pi k). \quad (34)$$

Repeating the same steps which led from equation (10) to equation (29), we obtain

$$Z_{\text{shifted}} = (-1)^{km} \int_0^{2\pi} \frac{d\lambda}{2\pi} e^{-im\lambda} \int \mathcal{D}(\text{fields})' |\psi_1^1(\mathbf{x})|^2 |\psi_2^1(\mathbf{y})|^2 e^{-S_0} \\ \times \exp \left[-i \frac{\lambda + \pi k}{4\pi} \int d^3x d^3y \left(\nabla \frac{1}{|\mathbf{x} - \mathbf{y}|} \times \mathbf{J}_2(\mathbf{y}) \right) \cdot \mathbf{J}_1(\mathbf{x}) \right]. \quad (35)$$

The integration over λ once again imposes constraint (32) in the partition function Z_{shifted} . As a consequence, it is easy to show that

$$Z_{\text{shifted}} = Z \quad (36)$$

for $k = 0, \pm 1, \pm 2, \dots$, because of the relation

$$e^{-i \frac{\pi k}{4\pi} \int d^3x d^3y (\nabla \frac{1}{|\mathbf{x} - \mathbf{y}|} \times \mathbf{J}_2(\mathbf{y})) \cdot \mathbf{J}_1(\mathbf{x})} = e^{-i\pi km} = (-1)^{km} \quad (37)$$

which is enforced by this constraint.

Thanks to the invariance under the shift (14) and equation (36), we are also able to prove the following identity:

$$\tilde{Z} \equiv \int_{-\infty}^{+\infty} \frac{d\lambda}{2\pi} e^{-im\lambda} Z(\lambda) = Z \quad (38)$$

where Z is the partition function given in (10) and $Z(\lambda)$ is defined in equation (11). The partition function \tilde{Z} differs from Z only by the domain of integration of the parameter λ , which in this case is the real line $(-\infty, +\infty)$. This is what one obtains if one derives the model of two closed polymers starting from two open polymers and then requiring that their ends coincide [6, 9].

To verify equation (38), we divide the domain of integration over λ into an infinite number of intervals $[2\pi l, 2\pi(l + 1)]$, where $-\infty \leq l \leq +\infty$ is an integer. In this way, \tilde{Z} becomes of the form:

$$\tilde{Z} = \sum_{l=-\infty}^{+\infty} \int_{2\pi l}^{2\pi(l+1)} \frac{d\lambda_l}{2\pi} e^{-im\lambda_l} Z(\lambda_l). \quad (39)$$

Since m is an integer, it is also possible to write

$$\tilde{Z} = \sum_{l=-\infty}^{+\infty} \int_0^{2\pi} \frac{d\lambda}{2\pi} e^{-im\lambda} Z(\lambda + 2\pi l). \quad (40)$$

Using equation (35) and constraint (32) it is now easy to show that the terms depending on l factorize as follows:

$$\tilde{Z} = \int \mathcal{D}(\text{fields})' |\psi_1^1(\mathbf{x})|^2 |\psi_2^1(\mathbf{y})|^2 e^{-S_0} \\ \times \delta \left(m - \frac{1}{4\pi} \int d^3x d^3y \mathbf{J}_1(\mathbf{x}) \cdot \left(\mathbf{J}_2(\mathbf{y}) \times \nabla \frac{1}{|\mathbf{x} - \mathbf{y}|} \right) \right) \sum_{l=-\infty}^{+\infty} e^{2\pi iml}. \quad (41)$$

Comparing the above expression with the expression of the partition function Z of equation (31) it is possible to conclude that $\tilde{Z} = Z$ apart from the infinite constant factor $\sum_{l=-\infty}^{+\infty} e^{2\pi iml} = \sum_{l=-\infty}^{+\infty} 1$, which does not change the physics of the problem.

5. The N -polymer model

In this section we consider the extension of the two-polymer model discussed above to the more realistic case of the fluctuations of N polymers. Let us denote with P_1, \dots, P_N the trajectories of the N polymers, which are constrained to satisfy the following relations:

$$\chi(P_i, P_j) = m_{ij} \quad \begin{matrix} m_{ij} = 0, \pm 1, \pm 2, \dots \\ i, j = 1, \dots, N. \end{matrix} \quad (42)$$

These conditions can be enforced in the partition function which describes the statistical mechanics of the N polymers by inserting the following product of Kronecker deltas:

$$\prod_{i=2}^N \prod_{\substack{j=1 \\ j < i}}^{N-1} \delta(\chi(P_i, P_j) - m_{ij}) = \prod_{i=2}^N \prod_{\substack{j=1 \\ j < i}}^{N-1} \int_0^{2\pi} \frac{d\lambda_{ij}}{2\pi} e^{-i\lambda_{ij}(m_{ij} - \chi(P_i, P_j))} \quad (43)$$

where the λ_{ij} are elements of an $N \times N$ matrix of Fourier parameters such that $\lambda_{ij} = 0$ whenever $j \geq i$ for $i, j = 1, \dots, N$. The field theoretical version of the N -polymer model has been derived in [9]. Its partition function is given by

$$\mathcal{Z}_N = \int \prod_{i=2}^N \prod_{\substack{j=1 \\ j < i}}^{N-1} \frac{d\lambda_{ij}}{2\pi} e^{-im_{ij}\lambda_{ij}} \mathcal{Z}_N(\lambda_{ij}) \quad (44)$$

where

$$\mathcal{Z}_N(\lambda_{ij}) = \int \mathcal{D}(\mathbf{A}\mathbf{B}\Psi_i\Psi_i^*) \prod_{i=1}^N \psi_i^1(\mathbf{x}^i) \psi_i^{*1}(\mathbf{y}^i) e^{-\mathcal{S}} \quad (45)$$

and the action \mathcal{S} is

$$\mathcal{S} = i\kappa \sum_{i=1}^{N-1} \int d^3x \mathbf{A}^i \cdot (\nabla \times \mathbf{B}^i) + \sum_{i=1}^N \int d^3x [|\nabla - i\mathbf{C}^i|\Psi_i|^2 + m_i^2|\Psi_i|^2]. \quad (46)$$

The fields $\Psi_i, \Psi_i^*, i = 1, \dots, N$, represent multiplets of replica fields:

$$\Psi_i = (\psi_i^1, \dots, \psi_i^{n_i}) \quad \Psi_i^* = (\psi_i^{*1}, \dots, \psi_i^{*n_i}) \quad (47)$$

and the vector fields \mathbf{C}^i are linear combinations of the Chern–Simons fields $\mathbf{A}^i, \mathbf{B}^i$:

$$\mathbf{C}^i = \sum_{j=1}^{i-1} \lambda_{ij} \mathbf{A}^j (1 - \delta_{i1}) + \mathbf{B}^i \delta_{i1}. \quad (48)$$

Finally,

$$\mathcal{D}(\mathbf{A}\mathbf{B}\Psi_i\Psi_i^*) = \prod_{i=1}^{N-1} \mathcal{D}\mathbf{A}^i \mathcal{D}\mathbf{B}^i \prod_{j=1}^N \prod_{a_j=1}^{n_j} \mathcal{D}\psi_j^{a_j} \mathcal{D}\psi_j^{*a_j}. \quad (49)$$

6. Elimination of the Chern–Simons fields from the N -polymer model

First of all, let us introduce auxiliary complex fields $\xi_i^{a_i}, \xi_i^{*a_i}$, where $i = 1, \dots, N$ and $1 \leq a_i \leq n_i$. In a similar way as we did in the case $N = 2$ in section 3, it is now possible to rewrite partition function (44) in the following way:

$$\mathcal{Z}_N(\lambda_{ij}) = \int \mathcal{D}(\mathbf{A}\mathbf{B}\Psi_i\Psi_i^*\xi_i\xi_i^*) e^{-\mathcal{S}_0 - \mathcal{S}_1(\lambda_{ij})} \quad (50)$$

where

$$\mathcal{D}(\mathbf{A}\mathbf{B}\Psi_i\Psi_i^*\xi_i\xi_i^*) = \prod_{i=1}^{N-1} \mathcal{D}\mathbf{A}^i \mathcal{D}\mathbf{B}^i \prod_{j=1}^N \prod_{a_j=1}^{n_j} \mathcal{D}\psi_j^{a_j} \mathcal{D}\psi_j^{*a_j} \mathcal{D}\xi_j^{a_j} \mathcal{D}\xi_j^{*a_j} \quad (51)$$

$$\mathcal{S}_0 = \sum_{i=1}^N \sum_{a_i=1}^{n_i} \int d^3x [-i(\nabla\psi_i^{*a_i} \cdot \xi_i^{a_i} + \nabla\psi_i^{a_i} \cdot \xi_i^{*a_i}) + \xi_i^{a_i} \cdot \xi_i^{*a_i}] \quad (52)$$

$$\mathcal{S}_1(\lambda_{ij}) = i\kappa \sum_{i=1}^{N-1} \int d^3x \mathbf{A}^i \cdot (\nabla \times \mathbf{B}^i) + i \sum_{i=1}^N \mathbf{C}^i \cdot \mathbf{J}_i \quad (53)$$

and

$$\mathbf{J}_i = \frac{1}{i} \sum_{a_i=1}^{n_i} [\psi_i^{*a_i} \xi_i^{a_i} + \psi_i^{a_i} \xi_i^{*a_i}]. \quad (54)$$

At this point we are ready to perform the integration over the fields $\mathbf{A}^i, \mathbf{B}^i$ in the partition function \mathcal{Z}_N . To this purpose, we need to evaluate the path integral:

$$\mathcal{Z}_{N\psi\xi} = \int \prod_{i=1}^N \mathcal{D}\mathbf{A}^i \mathcal{D}\mathbf{B}^i e^{-\mathcal{S}_i(\lambda_{ij})}. \quad (55)$$

One finds after a first integration over the fields \mathbf{B}^j :

$$\mathcal{Z}_{N\psi\xi} = \int \prod_{i=1}^N \mathcal{D}\mathbf{A}^i e^{-i\kappa \sum_{i=2}^N \sum_{j=1}^{i-1} \lambda_{ij} \mathbf{A}^j \cdot \mathbf{J}_i} \prod_{i=1}^{N-1} \delta(\kappa \nabla \times \mathbf{A}^i - \mathbf{J}_i). \quad (56)$$

The product of Dirac δ -functions in (56) enforces the constraints:

$$\kappa \nabla \times \mathbf{A}^i = \mathbf{J}_i \quad (57)$$

whose solution (in components) is

$$A_\mu^i = \frac{1}{4\pi\kappa} \int d^3y \epsilon_{\mu\nu\rho} \frac{(x-y)^\nu}{|\mathbf{x}-\mathbf{y}|^3} J_j^\rho(\mathbf{y}) \quad \mu, \nu, \rho = 1, 2, 3. \quad (58)$$

It is now easy to show that

$$\mathcal{Z}_{N\psi\xi} = \exp \left\{ -\frac{i}{4\pi} \sum_{i=2}^N \sum_{j=1}^{i-1} \int d^3x d^3y \lambda_{ij} \epsilon_{\mu\nu\rho} \frac{(x-y)^\nu}{|\mathbf{x}-\mathbf{y}|^3} J_i^\mu(\mathbf{x}) J_j^\rho(\mathbf{y}) \right\}. \quad (59)$$

Substituting this result into the expression of the partition function \mathcal{Z}_N in equation (44) one obtains

$$\begin{aligned} \mathcal{Z}_N &= \int \prod_{i=2}^N \prod_{\substack{j=1 \\ j < i}}^{N-1} \frac{d\lambda_{ij}}{2\pi} e^{-im_{ij}\lambda_{ij}} \prod_{i=1}^N \prod_{a_i=1}^{n_i} \mathcal{D}\psi_i^{a_i} \mathcal{D}\psi_i^{*a_i} \mathcal{D}\xi_i^{a_i} \mathcal{D}\xi_i^{*a_i} \\ &\times \exp \left\{ -\mathcal{S}_0 + \frac{i}{4\pi} \sum_{i=2}^N \sum_{j=1}^{i-1} \int d^3x d^3y \lambda_{ij} \mathbf{J}_i(\mathbf{x}) \cdot \left[\nabla \frac{1}{|\mathbf{x}-\mathbf{y}|} \times \mathbf{J}_j(\mathbf{y}) \right] \right\}. \quad (60) \end{aligned}$$

From the formulation of the partition function \mathcal{Z}_N given by equation (60) it turns out that the Fourier variables λ_{ij} play the role of Lagrange multipliers imposing the constraints:

$$m_{ij} = \frac{1}{4\pi} \int d^3x d^3y \mathbf{J}_i(\mathbf{x}) \cdot \left[\nabla \frac{1}{|\mathbf{x}-\mathbf{y}|} \times \mathbf{J}_j(\mathbf{y}) \right] \quad (61)$$

which are the generalization to the N -polymer case of condition (32). Once again, in the limit in which all polymers are disentangled, the effects of the topological interactions do not disappear. The invariance of the partition function \mathcal{Z}_N with respect to the shifts:

$$\lambda_{ij} \rightarrow \lambda_{ij} + \pi k_{ij} \quad k_{ij} = 0, \pm 1, \pm 2, \dots \quad (62)$$

can also be proved using similar methods to those employed in section 4.

7. Conclusions

In this paper we have studied Abelian multi-Chern–Simons field theories coupled with matter fields. Attention has been concentrated on those models which are relevant to the statistical mechanics of polymers, but some results are valid for any theory containing Chern–Simons fields. Motivated by the difficulties which one encounters when dealing with these theories due to the presence of the topological fields and their kinetic terms, we have proposed a procedure to eliminate these fields from the action. The advantage is that now the partition function explicitly depends on the topological numbers m_{ij} and the polymer action is polynomial in the fields, in contrast to the action derived in [9]. The price to be paid is that the new action contains auxiliary vector fields and has a nonlocal two-body interaction.

With the help of the new formulation it has been possible to show that action (46) is invariant under the shifts of the coupling constants λ_{ij} given in equation (62). This symmetry, which was not *a priori* evident in action (46), has been used in order to show the equivalence of all models of topologically linked polymers differing by the range of integration of the Fourier variables λ_{ij} , see equations (36) and (38). The generalization of these results to any N starting from the partition function of equation (60) is straightforward. Finally, it has been clarified that the parameters λ_{ij} in polymer models are Lagrange multipliers, which impose the constraints (61). These conditions represent clearly the field theoretical version of the topological constraints (6), but are not topological relations by themselves. This is a natural consequence of the fact that, in the process of elimination of the Chern–Simons fields, topological and non-topological terms have been mixed together.

To conclude, we would like to address some problems which are still open. First of all, experiments suggest that in the presence of topological constraints there are attractive forces acting on polymers [19]. In particular, there is evidence that the strength of these forces increases with increasing complexity of the topological configuration of the system. A perturbative calculation at the one-loop approximation confirms the presence of such forces in the two-polymer model [15], but it is difficult to estimate how their strength depends on m starting from the partition function (31). Another open question is how the phenomenon of confinement that is active in multi-Chern–Simons field theories may influence the statistical behaviour of the polymers.

Appendix. Hubbard–Stratonovich transformations

In deriving equations (15)–(17), as well as equations (50)–(53), we have used the generalization to a path integral of the following Gaussian integral formula:

$$\begin{aligned} \int \prod_{\alpha=1}^n dz_{\alpha} \bar{z}_{\alpha} \exp \left[- \sum_{\alpha, \beta=1}^n \bar{z}_{\alpha} A_{\alpha\beta} z_{\beta} + i \sum_{\alpha=1}^n (\bar{b}_{\alpha} z_{\alpha} + b_{\alpha} \bar{z}_{\alpha}) \right] \\ = (2\pi i)^n (\det A)^{-1} \exp \sum_{\alpha, \beta=1}^n [-\bar{b}_{\alpha} (A^{-1})_{\alpha\beta} b_{\beta}]. \end{aligned} \quad (A.1)$$

In the above equation z_α, \bar{z}_α are a set of complex variables and b_α, \bar{b}_α are constants. To prove equation (A.1) it is sufficient to perform the transformation

$$z_\alpha = z'_\alpha + i(A^{-1})_{\alpha\beta} b_\beta \quad (\text{A.2})$$

$$\bar{z}_\alpha = \bar{z}'_\alpha + i\bar{b}_\beta (A^{-1})_{\alpha\beta} \quad (\text{A.3})$$

which are the analogues of equations (23), (24). The Gaussian formula (A.1) can be brought into an equivalent form after switching to real components x_1, \dots, x_{2n} and c_1, \dots, c_{2n} of z_α and b_α , respectively:

$$z_\alpha = x_\alpha + ix_{n+\alpha} \quad \bar{z}_\alpha = x_\alpha - ix_{n+\alpha} \quad (\text{A.4})$$

$$b_\alpha = c_\alpha + ic_{n+\alpha} \quad \bar{b}_\alpha = c_\alpha - ic_{n+\alpha}. \quad (\text{A.5})$$

Substituting equations (A.4), (A.5) in (A.1) one obtains the following Gaussian identity, which is familiar in polymer physics because it is used to simplify the excluded volume interactions:

$$\int \prod_{j=1}^{2n} [dx_j] \exp \left(- \sum_{j=1}^{2n} x_j^2 + 2i \sum_{j=1}^{2n} c_j x_j \right) = (2\pi)^n \exp \left(-2 \sum_{j=1}^{2n} c_j^2 \right). \quad (\text{A.6})$$

References

- [1] Wen X G and Zee A 1992 *Phys. Rev. B* **46** 2290
- [2] Frohlich J and Zee A 1991 *Nucl. Phys. B* **364** 517
- [3] Lee D H and Fisher M P 1989 *Phys. Rev. Lett.* **63** 903
- [4] Birmingham D, Blau M, Rakowski M and Thompson G 1991 *Phys. Rep.* **209** 129
- [5] Wesolowski D, Hosotani Y and Ho C L 1994 *Int. J. Mod. Phys. A* **9** 969 (*Preprint hep-th/9302121*)
- [6] Ferrari F 2002 *Ann. Phys., Lpz.* **11** 255
- [7] Cornalba L and Wilczek F 1997 *Phys. Rev. Lett.* **78** 4679 (*Preprint hep-th/9703131*)
- [8] de Wild Propitius M 1997 *Phys. Lett. B* **410** 188 (*Preprint hep-th/9704063*)
- [9] Ferrari F, Kleinert H and Lazzizzera I 2000 *Int. J. Mod. Phys. B* **14** 3881 (*Preprint cond-mat/0005300*)
- [10] Brereton M G and Vilgis T A 1995 *J. Phys. A: Math. Gen.* **28** 1149
- [11] Tanaka F 1982 *Prog. Theor. Phys.* **68** 148
- [12] Chaichian M and Chen W F 1998 *Phys. Rev. D* **58** 125004 (*Preprint hep-th/9806127*)
- [13] Bietenholz W, Nishimura J and Sodano P 2002 *Preprint hep-lat/0207010*
- [14] Berruto F, Diamantini M C and Sodano P 2000 *Phys. Lett. B* **487** 366 (*Preprint hep-th/0004203*)
- [15] Ferrari F and Lazzizzera I 1999 *Nucl. Phys. B* **559** 673
- [16] Ferrari F and Lazzizzera I 1998 *Phys. Lett. B* **444** 167
- [17] Vilgis T A 2000 *Phys. Rep.* **336** 167
- [18] Kleinert H 1995 *Path Integrals* 2nd edn (Singapore: World Scientific)
- [19] Levene S D, Donahue C, Boles T C and Cozzarelli N R 1995 *Biophys. J.* **69** 1036